First- and second-order forces on a cylinder submerged under a free surface

By T. FRANCIS OGILVIE

U.S. Office of Naval Research, London Branch Office

Received 2 October 1962 and in revised form 15 January 1963

1. Introduction

Several years ago Dean (1948) showed how to solve the linearized potential problem of water waves passing over a submerged circular cylinder. He discovered the remarkable fact that there is no reflexion from the cylinder; the transmitted waves have the same amplitude as the incident waves, but they suffer a phase shift in passing the cylinder. Soon after the publication of Dean's paper, Ursell (1950) investigated the problem anew. He placed the solution on a rigorous basis, supplied a uniqueness proof, and developed a form of the solution with which it was reasonable to perform calculations.

Ursell's procedure is here applied and extended to several specific problems. In particular, the first-order oscillatory force and the second-order steady force are calculated for the following situations: (*a*) the cylinder is restrained from moving under the effect of incident sinusoidal waves; (*b*) the cylinder is forced to oscillate sinusoidally in otherwise calm water; (*c*) the cylinder, which is neutrally buoyant, is allowed to respond to the first-order oscillatory forces. In all cases the problem is treated by two-dimensional methods. The water is considered to be infinitely deep.

It is first proved that knowledge of the first-order potential supplies information sufficient to solve these problems. The solutions are obtained and then numerical results are presented, the relevant quantities being plotted as functions of depth, with wave-number as a parameter. Although the main purpose here is to calculate these forces, it is a simple matter also to extend the results of Dean and Ursell on the transmission of waves. It is found that in situation (*c*), as in (*a*), there are no reflected waves. For both (*a*) and (*c*), curves are presented for the phase shift of the transmitted wave. In case (*b*) it is shown that outgoing waves are generated in one direction only, if the cylinder centre follows a circular orbit.

2. The second-order problem

Assume that a circular cylinder is located under a free surface, with its centre at $x = \xi(t)$, $y = -h+\eta(t)$. The instantaneous surface of the body is then specified by

$$S(x,y,t) = [x-\xi(t)]^2+[y+h-\eta(t)]^2-a^2 = 0, \tag{1}$$

where a is the radius of the cylinder. We also define a surface S_0:

$$S_0(x,y) = x^2+(y+h)^2-a^2 = 0 \tag{1'}$$

(see figure 1). The undisturbed free surface is taken as the x-axis, and the instantaneous free surface will be specified by

$$y - Y(x,t) = 0. \tag{2}$$

The y-axis is positive upwards.

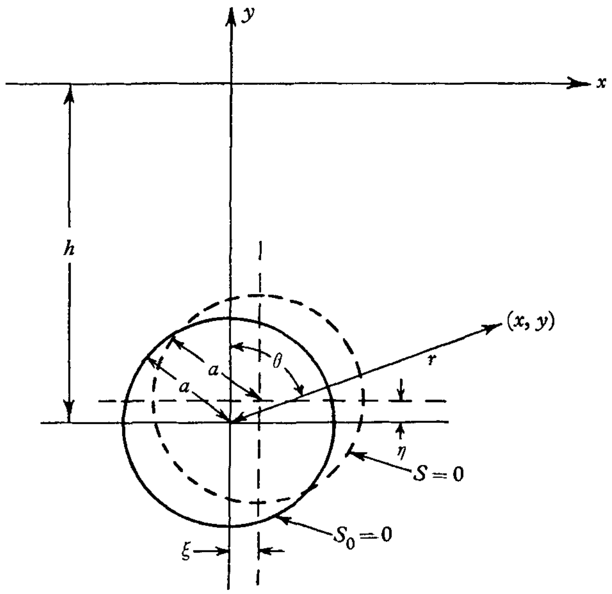

FIGURE 1. Geometry of the problem.

We seek a velocity potential,† $\Phi(x,y,t)$, which satisfies (see Stoker 1957, or Wehausen & Laitone 1960)

$$\Phi_{xx} + \Phi_{yy} = 0, \tag{3}$$

$$\Phi_x Y_x - \Phi_y + Y_t = 0 \quad \text{on} \quad y = Y(x,t), \tag{4}$$

$$gY + \Phi_t + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2) = 0 \quad \text{on} \quad y = Y(x,t), \tag{5}$$

$$\Phi_x S_x + \Phi_y S_y + S_t = 0 \quad \text{on} \quad S = 0. \tag{6}$$

Equation (3) must hold for all time, t, and for all (x,y) in the fluid domain, i.e. for $y < Y(x,t)$ and $S(x,y,t) > 0$. Equation (5) is, of course, Bernoulli's equation, and (4) and (6) express the usual kinematic condition at a boundary of a perfect fluid. There will also be conditions at infinity, which will be formulated later.

We assume that all of the dependent variables can be expressed in power series in terms of some small parameter, ϵ:

$$\left.\begin{aligned} \Phi(x,y,t) &= \epsilon\Phi^{(1)}(x,y,t) + \epsilon^2\Phi^{(2)}(x,y,t) + \ldots, \\ Y(x,t) &= \epsilon Y^{(1)}(x,t) + \epsilon^2 Y^{(2)}(x,t) + \ldots, \\ \xi(t) &= \epsilon\xi^{(1)}(t) + \epsilon^2\xi^{2}(t) + \ldots, \\ \eta(t) &= \epsilon\eta^{(1)}(t) + \epsilon^2\eta^{(2)}(t) + \ldots. \end{aligned}\right\} \tag{7}$$

† We choose the potential so that its positive gradient is the velocity.

(The superscripts in parentheses are strictly indices.) It is then assumed that the potential function can be continued analytically into the region $y < 0$, $S_0 > 0$, and that the value of $\Phi(x,y,t)$ (and values of its derivatives) on the two boundaries can be expressed in terms of Taylor series about the undisturbed positions of these boundaries. The following conditions are found:

$$\Phi_{xx}^{(p)} + \Phi_{yy}^{(p)} = 0 \quad \text{for} \quad y < 0,\ S_0 > 0 \quad (p = 1, 2, 3, \ldots); \tag{8}$$

$$\Phi_{tt}^{(1)} + g\Phi_{y}^{(1)} = 0 \quad \text{on} \quad y = 0, \tag{9a}$$

$$\Phi_{tt}^{(2)} + g\Phi_{y}^{(2)} = -2\Phi_x^{(1)}\Phi_{xt}^{(1)} - 2\Phi_y^{(1)}\Phi_{yt}^{(1)} + g^{-1}\Phi_t^{(1)}\Phi_{tty}^{(1)} + \Phi_t^{(1)}\Phi_{yy}^{(1)} \quad \text{on} \quad y = 0, \tag{9b}$$

etc.;

$$\mathbf{r}\,.\,[\nabla\Phi^{(1)} - \dot{\boldsymbol{\zeta}}^{(1)}(t)] = 0 \quad \text{on} \quad S_0 = 0, \tag{10a}$$

$$\mathbf{r}\,.\,[\nabla\Phi^{(2)} - \dot{\boldsymbol{\zeta}}^{(2)}(t)] = -(\mathbf{r}\,.\,\nabla)\,[\boldsymbol{\zeta}^{(1)}(t)\,.\,\nabla\Phi^{(1)}] + \boldsymbol{\zeta}^{(1)}(t)\,.\,[\nabla\Phi^{(1)} - \dot{\boldsymbol{\zeta}}^{(1)}(t)] \quad \text{on} \quad S_0 = 0, \tag{10b}$$

etc.; we have written

$$\mathbf{r} = x\mathbf{i} + (y+h)\,\mathbf{j}, \tag{11a}$$

$$\boldsymbol{\zeta}^{(p)}(t) = \xi^{(p)}(t)\,\mathbf{i} + \eta^{(p)}(t)\,\mathbf{j}, \tag{11b}$$

with $\mathbf{i}$ and $\mathbf{j}$ unit vectors along the x- and y-axes, respectively.

If one finds a potential function, $\Phi^{(1)}(x,y,t)$, which satisfies equations (9a) and (10a), then the right-hand sides of equations (9b) and (10b) are known. Similarly, further conditions could be found on $\Phi^{(p)}(x,y,t)$ and $\boldsymbol{\zeta}^{(p)}(t)$, for $p > 2$, in each pair of which the left-hand sides would be of the form of (9a) and (10a) and the right-hand sides would depend only on the lower-order solutions.

To find the force on the cylinder, we integrate the pressure, $p(x,y,t)$, around the boundary in the clockwise direction:

$$X(t) - iY(t) = i\int_{S=0} p(x,y,t)\,(dx - i\,dy).$$

Let us define a system of polar co-ordinates, (r,θ):

$$x = r\sin\theta, \quad y = -h + r\cos\theta. \tag{12a, b}$$

Every point (x,y) on $S = 0$ can be referred to a point on $S_0 = 0$; that is, if (x,y) is on $S = 0$, then

$$x = \xi + a\sin\theta, \quad y = -h + \eta + a\cos\theta,$$

where clearly $(a\sin\theta,\ -h + a\cos\theta)$ is a point on $S_0 = 0$. These relations are equivalent to

$$x - iy = ih + (\xi - i\eta) - ia\,e^{i\theta},$$

so that, as the variable of integration follows the circle $S = 0$, we have

$$dx - i\,dy = a\,e^{i\theta}\,d\theta.$$

The potentials to be found will be harmonic in the lower half-plane, outside a certain circle with centre at $(0, -h)$. In fact, it can be shown that, if $\xi \equiv \eta \equiv 0$, the potentials are harmonic outside $x^2 + (y+h)^2 = (h-l)^2$, where l is the length of the tangent from $(0,0)$ to the circle $x^2 + (y+h)^2 = a^2$. Clearly, if $h - a > 0$, then $h - l < a$. If the motions of the cylinder are sufficiently small, a similar result can again be proved, and the potentials can be expanded in Taylor series in a finite neighbourhood about every point of S or S_0. The same must then be true of the pressure functions, so that

$$\begin{aligned} p(x,y,t)|_{S=0} &= p(\xi + a\sin\theta,\ -h+\eta+a\cos\theta, t) \\ &= p(a\sin\theta,\ -h+a\cos\theta,t)+\xi(t)\,p_x(a\sin\theta,\ -h+a\cos\theta,t) \\ &\qquad + \eta(t)\,p_y(a\sin\theta,\ -h+a\cos\theta,t)+\ldots \\ &= [p+\xi(t)\,p_x+\eta(t)\,p_y+\ldots]_{S_0=0}. \end{aligned}$$

We now have for the force on the cylinder

$$X(t)-iY(t) = ia\int_{-\pi}^{\pi} e^{i\theta}\,[p+\xi(t)\,p_x+\eta(t)\,p_y+\ldots]_{S_0=0}\,d\theta. \tag{13}$$

To the expansions in (7), we add another:

$$p(x,y,t) = \epsilon p^{(1)}(x,y,t)+\epsilon^2 p^{(2)}(x,y,t)+\ldots, \tag{14}$$

where $p(x,y,t)$ is the hydrodynamic pressure (the difference between the actual pressure and the hydrostatic pressure measured from $y = 0$). From Bernoulli's equation and from (7),

$$\begin{aligned} p(x,y,t) &= -\rho\Phi_t - \tfrac{1}{2}\rho[\Phi_x^2+\Phi_y^2] \\ &= \epsilon[-\rho\Phi_t^{(1)}]+\epsilon^2[-\rho\Phi_t^{(2)}-\tfrac{1}{2}\rho(\Phi_x^{(1)})^2-\tfrac{1}{2}\rho(\Phi_y^{(1)})^2]+O(\epsilon^3). \end{aligned}$$

Also, $\quad p_x(x,y,t) = \epsilon[-\rho\Phi_{xt}^{(1)}]+O(\epsilon^2), \quad p_y(x,y,t) = \epsilon[-\rho\Phi_{yt}^{(1)}]+O(\epsilon^2).$

When these relations, as well as (7), are used in (13), we obtain

$$X(t)-iY(t) = \epsilon[X^{(1)}(t)-iY^{(1)}(t)]+\epsilon^2[X^{(2)}(t)-iY^{(2)}(t)]+O(\epsilon^3),$$

where

$$X^{(1)}(t)-iY^{(1)}(t) = -ia\rho\int_{-\pi}^{\pi} e^{i\theta}\,\Phi_t^{(1)}|_{r=a}\,d\theta; \tag{15}$$

$$X^{(2)}(t)-iY^{(2)}(t) = -ia\rho\int_{-\pi}^{\pi} e^{i\theta}\,[\Phi_t^{(2)}+\tfrac{1}{2}(\Phi_x^{(1)})^2+\tfrac{1}{2}(\Phi_y^{(1)})^2+\xi^{(1)}\Phi_{xt}^{(1)}+\eta^{(1)}\Phi_{yt}^{(1)}]_{r=a}\,d\theta. \tag{16}$$

In the problem to be considered presently, $\Phi^{(1)}(x,y,t)$ will vary sinusoidally in time. From (9*b*) and (10*b*), it is clear then that $\Phi^{(2)}(x,y,t)$ will have two components: (1) a time-independent ('d.c.') part and (2) a part that oscillates sinusoidally at twice the frequency of the first-order solution. If we calculate the time-average of $X^{(2)}(t)-iY^{(2)}(t)$, using (16), we see that $\Phi^{(2)}(x,y,t)$ does not contribute at all. Thus†

$$\overline{X^{(2)}(t)}^t-i\overline{Y^{(2)}(t)}^t = -ia\rho\int_{-\pi}^{\pi} e^{i\theta}\{\tfrac{1}{2}\overline{(\Phi_x^{(1)})^2}^t+\tfrac{1}{2}\overline{(\Phi_y^{(1)})^2}^t+\overline{\xi^{(1)}\Phi_{xt}^{(1)}}^t+\overline{\eta^{(1)}\Phi_{yt}^{(1)}}^t\}|_{r=a}\,d\theta. \tag{16'}$$

If we let $\Phi^{(1)}(x,y,t)$ be the real part of a function of a complex variable, say $f(z,t)$, where $z = x+iy$, then

$$\overline{X^{(2)}(t)}^t-i\overline{Y^{(2)}(t)}^t = -ia\rho\int_{-\pi}^{\pi} e^{i\theta}\,[\tfrac{1}{2}\overline{f'(z,t)\,\overline{f'(z,t)}}^t+\mathrm{Re}\,\overline{\{(\xi^{(1)}+i\eta^{(1)})\,\partial f'(z,t)/\partial t\}}^t]_{r=a}\,d\theta. \tag{16''}$$

The prime denotes differentiation with respect to z.

† An over bar with a t following it will be used to indicate *time averages*. A bar without index will indicate *complex conjugate*.

3. General solution of the first-order problem

The first-order problem has been entirely solved by Ursell and this section contains only a restatement of some of his results which will be needed later.

Except for the possible presence of incident sinusoidal waves coming from infinity, it is assumed that the only disturbance of the free surface far from the cylinder must appear as outgoing waves. The pulsating-singularity potentials in Appendix A, equations (56), all satisfy the free-surface condition, ($9a$), and also represent outgoing waves as $x \to \pm\infty$. On $r = a$ they form a complete set in terms of which the normal fluid velocity can be expanded (provided no fluid is generated inside the cylinder). So it is necessary only to combine them linearly in such a way that ($10a$) is satisfied.

Let the only external hydrodynamic disturbance be an incident wave, the potential for which is the real part of

$$f_0(z,t) = A\exp\left[-i(\nu z+\sigma t)\right] = A\exp\left[-\nu h\right]\exp\left[\nu r\, e^{-i\theta}\right]\exp\left[-i\sigma t\right], \tag{17}$$

where $\nu = \sigma^2/g$, and A is a real constant. [The elevation of the free surface in the incident wave is then

$$-\frac{1}{g}\frac{\partial}{\partial t}\operatorname{Re}\{f_0(z,t)\}|_{y=0} = \frac{A\sigma}{g}\sin\left(\nu x+\sigma t\right).$$

Thus the amplitude of the incident free surface wave is $H_0 = A\sigma/g$.] Let $\xi^{(1)}(t)$ and $\eta^{(1)}(t)$ vary sinusoidally also, either in response to the wave action or as a result of some force applied directly to the cylinder. Specifically, let

$$\xi^{(1)}(t) = \xi_1\sin\sigma t+\xi_2\cos\sigma t, \quad \eta^{(1)}(t) = \eta_1\sin\sigma t+\eta_2\cos\sigma t. \tag{18}$$

Then the point on the cylinder at the angle θ has an outward radial velocity (to first order)

$$\sigma\cos\theta(\eta_1\cos\sigma t-\eta_2\sin\sigma t)+\sigma\sin\theta(\xi_1\cos\sigma t-\xi_2\sin\sigma t). \tag{18'}$$

Let the entire first-order potential be the real part of

$$f(z,t) = f_0(z,t)+\sum_{n=1}^{\infty}\{\alpha_n f_{n1}(z,t)+\beta_n f_{n2}(z,t)+\gamma_n g_{n1}(z,t)+\delta_n g_{n2}(z,t)\}. \tag{19}$$

The functions f_{n1}, f_{n2}, g_{n1}, and g_{n2} are given in Appendix A. By ($10a$),

$$\partial\operatorname{Re}\{f(z,t)\}/\partial r$$

equals the expressions ($18'$). Thus the following sets of equations are obtained:

$$\alpha_m+(\nu a)^{2m}\sum_{n=1}^{\infty}\frac{(m+n)!}{m!\,(n-1)!}(A_{m+n}\alpha_n-B_{m+n}\beta_n) = \frac{\sigma}{\nu}\eta_2(\nu a)^2\delta_{m1}, \tag{20a}$$

$$\beta_m+(\nu a)^{2m}\sum_{n=1}^{\infty}\frac{(m+n)!}{m!\,(n-1)!}(A_{m+n}\beta_n+B_{m+n}\alpha_n) = -\frac{\sigma}{\nu}\eta_1(\nu a)^2\delta_{m1}+\frac{A\,e^{-\nu h}(\nu a)^{2m}}{m!}, \tag{20b}$$

$$\gamma_m+(\nu a)^{2m}\sum_{n=1}^{\infty}\frac{(m+n)!}{m!\,(n-1)!}(A_{m+n}\gamma_n-B_{m+n}\delta_n) = -\frac{\sigma}{\nu}\xi_2(\nu a)^2\delta_{m1}+\frac{A\,e^{-\nu h}(\nu a)^{2m}}{m!}, \tag{20c}$$

$$\delta_m+(\nu a)^{2m}\sum_{n=1}^{\infty}\frac{(m+n)!}{m!\,(n-1)!}(A_{m+n}\delta_n+B_{m+n}\gamma_n) = \frac{\sigma}{\nu}\xi_1(\nu a)^2\delta_{m1}, \tag{20d}$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

and the A_j and B_j are known constants, given in Appendix A by equations (55). Thus we have four infinite sets of algebraic equations for the four infinite sets of coefficients, α_m, β_m, γ_m, δ_m. The conditions under which solutions exist were discussed by Ursell and will not be considered here. We shall assume throughout that unique solutions exist. The method for solving these equations, and, in particular, for uncoupling them, was also discussed by Ursell. Further mention of this problem will be found in Appendix B.

If the formulae (56) of Appendix A are substituted into (19) and the order of summations is reversed,† (20) can be used to eliminate the double sums, and the complex potential becomes:

$$\begin{aligned} f(z,t) = \cos\sigma t\Bigg\{&\sum_1^\infty \frac{(\beta_m + i\delta_m)\, e^{im\theta}}{(\nu r)^m} + \sum_1^\infty \frac{(\beta_m - i\delta_m)\,(\nu r)^m\, e^{-im\theta}}{(\nu a)^{2m}} \\ &+ \sigma r\, e^{-i\theta}(\eta_1 + i\xi_1) - \sum_1^\infty n[A_n(\beta_n - i\delta_n) + B_n(\alpha_n - i\gamma_n)]\Bigg\} \\ +\sin\sigma t\Bigg\{&\sum_1^\infty \frac{(\alpha_m + i\gamma_m)\, e^{im\theta}}{(\nu r)^m} + \sum_1^\infty \frac{(\alpha_m - i\gamma_m)\,(\nu r)^m\, e^{-im\theta}}{(\nu a)^{2m}} \\ &- \sigma r\, e^{-i\theta}(\eta_2 + i\xi_2) - \sum_1^\infty n[A_n(\alpha_n - i\gamma_n) - B_n(\beta_n - i\delta_n)]\Bigg\}. \end{aligned} \quad (19')$$

The last sum in each bracket is a constant which does not affect any of the subsequent force calculations. The constant A does not appear explicitly now, but the values of α_m, β_m, γ_m and δ_m all depend on it, as well as on ξ_1, ξ_2, η_1, η_2.

At this point, the problem is solved, in principle.

(*a*) If the cylinder is restrained from moving, then $\xi_1 = \xi_2 = \eta_1 = \eta_2 = 0$. Equations (20) yield the coefficients for $f(z,t)$, as defined in (19). The desired forces are then given by (15) and (16″).

(*b*) If there are no incident waves and the cylinder is forced to oscillate with given amplitude, direction, and frequency, then ξ_1, ξ_2, η_1 and η_2 are known, and $A = 0$. The coefficients are obtained from solution of (20), and the force components are obtained from (15) and (16″).

(*c*) If the cylinder is assumed neutrally buoyant, equations (20) give the coefficients in terms of the unknown motion parameters, ξ_1, ξ_2, η_1, η_2, and then the motion is determined by solution of the equations of motion of a rigid body, with the force again given by (15). Then the coefficients are completely known and (16″) gives the second-order steady force.

It is of interest to carry these solutions further. The next three sections treat the above cases in more detail.

† The singularity potentials in Appendix A are expressed by Laurent series. It is legitimate to add sums of such series term-by-term if the resulting series converge uniformly. But the potential is itself analytic and single-valued in $a < |z + ih| < 2h - a$. So it must have a Laurent expansion, which must then be (19′).

4. Cylinder restrained

We assume here that the cylinder is rigidly held in place while waves pass by. That is, $\xi_1 = \xi_2 = \eta_1 = \eta_2 = 0$, and $f_0(z,t)$ is given by (17). From (20) it is clear that

$$\gamma_m = \beta_m, \quad \delta_m = -\alpha_m. \tag{21}$$

Thus, from (19′),

$$\begin{aligned} f(z,t) &\equiv f_1(z,t) \\ &= -i\,e^{i\sigma t}\sum_1^\infty \frac{(\alpha_m+i\beta_m)\,e^{im\theta}}{(\nu r)^m} + i\,e^{-i\sigma t}\sum_1^\infty \frac{(\alpha_m-i\beta_m)\,(\nu r)^m\,e^{-im\theta}}{(\nu a)^{2m}} \\ &\qquad - i\,e^{-i\sigma t}\sum_1^\infty m(A_m-iB_m)\,(\alpha_m-i\beta_m). \end{aligned} \tag{22}$$

The first-order force is found from (15):

$$X_1^{(1)}(t) - i\,Y_1^{(1)}(t) = -(2\pi\rho\sigma/\nu)\,[(\alpha_1\sin\sigma t+\beta_1\cos\sigma t) - i(\beta_1\sin\sigma t-\alpha_1\cos\sigma t)]. \tag{23}$$

The second-order steady force is found from (16″):

$$\begin{aligned} &\overline{X_1^{(2)}(t)}^t - i\,\overline{Y_1^{(2)}(t)}^t \\ &\quad = -2i\pi\rho\nu\sum_1^\infty \frac{m(m+1)\,[(\alpha_m\alpha_{m+1}+\beta_m\beta_{m+1}) + i(\beta_m\alpha_{m+1}-\alpha_m\beta_{m+1})]}{(\nu a^2)^{2m+2}}. \end{aligned} \tag{24}$$

In Appendix B it is shown that, if ϵ_m satisfies

$$\epsilon_m + (\nu a)^{2m}\sum_{n=1}^\infty \frac{(m+n)!}{m!\,(n-1)!}A_{m+n}\epsilon_n = \frac{(\nu a)^{2m}}{m!}, \tag{25}$$

and if

$$S_\epsilon = 2\pi\,e^{-2\nu h}\sum_1^\infty \frac{\epsilon_n}{(n-1)!}, \tag{25′}$$

then

$$\alpha_m = A\,e^{-\nu h}S_\epsilon\epsilon_m/(1+S_\epsilon^2), \quad \beta_m = A\,e^{-\nu h}\epsilon_m/(1+S_\epsilon^2). \tag{26a, b}$$

These results can be used to simplify (23) and (24). We note again that $H_0 = A\sigma/g$ and also that $H_0^2 = \nu A^2/g$, where H_0 is the amplitude of the incident surface wave. Then, from (23),

$$X_1^{(1)}(t) - i\,Y_1^{(1)}(t) = -(2\pi\rho g H_0/\nu)\,\{\epsilon_1 e^{-\nu h}/(1+S_\epsilon^2)^{\frac{1}{2}}\}\exp\{-i(\sigma t-\psi_1)\}, \tag{27}$$

where

$$\psi_1 = \tan^{-1}S_\epsilon. \tag{27′}$$

From (24),

$$\overline{X_1^{(2)}(t)}^t = 0; \quad \overline{Y_1^{(2)}(t)}^t = \frac{2\pi\rho g H_0^2\,e^{-2\nu h}}{1+S_\epsilon^2}\sum_1^\infty \frac{m(m+1)\,\epsilon_m\epsilon_{m+1}}{(\nu a)^{2m+2}}. \tag{28a, b}$$

Numerical results from equations (27) and (28) are presented as the solid curves in figures 2, 3 and 4. The curve for each value of νa terminates at the left where $2\nu h = 2\nu a$. An extension beyond such an abscissa would have no physical meaning, because the cylinder would not be completely submerged. In fact, the part of each curve near the left-hand end must be interpreted with considerable care. The amplitude of the incident waves must be much less than the clearance between the top of the cylinder and the undisturbed free surface if the linearized theory is to have meaning.

If we take the undisturbed incident potential wave (see equation (17)) as a reference, we see from figure 3 that the phase of the oscillatory force lags more and more as the cylinder is considered to be closer and closer to the surface. An interpretation of this situation will be presented later when transmission of the wave is considered.

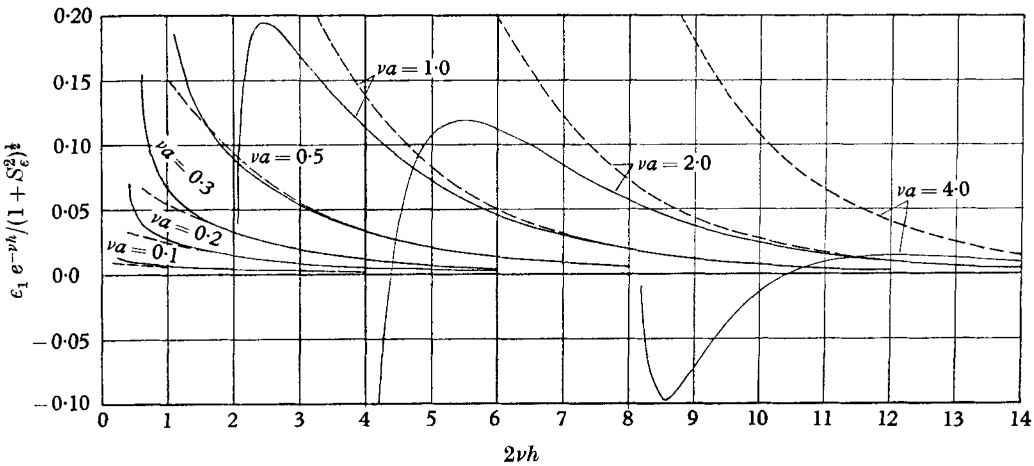

FIGURE 2. Amplitude of oscillatory force on restrained cylinder.

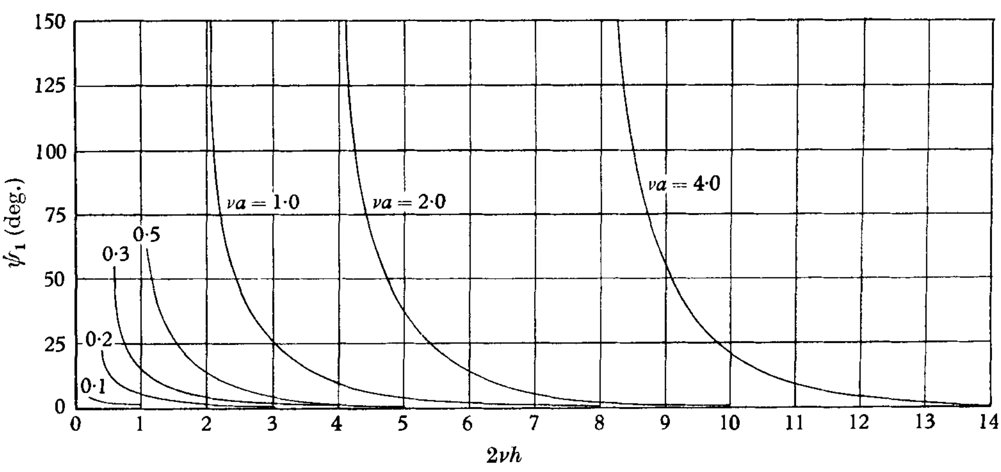

FIGURE 3. Phase lag of oscillatory force on restrained cylinder.

It is of interest to estimate the results for $\nu a \ll 1$. Let

$$\epsilon_m = \frac{(\nu a)^{2m}}{m!} \sum_{\mu=0}^{\infty} \epsilon_{m\mu} (\nu a)^{2\mu}. \tag{29}$$

This expansion is justified by Ursell. Also, let

$$\gamma_{mn} = \{(m+n)!/(n-1)!\} A_{m+n}. \tag{30}$$

Note that A_{m+n} (and thus γ_{mn}) is a function only of $2\nu h$. For the forces we have approximately

$$X_1^{(1)}(t) - iY_1^{(1)}(t) = -(2\pi\rho g H_0/\nu)(\nu a)^2 e^{-\nu h}$$
$$\times\{[1-\gamma_{11}(\nu a)^2 + (\gamma_{11}^2 - \tfrac{1}{2}\gamma_{12} - 4\pi^2 e^{-4\nu h})(\nu a)^4 + \dots]$$
$$+ 2\pi i\, e^{-2\nu h}(\nu a)^2[1 + (\tfrac{1}{2} - 2\gamma_{11})(\nu a)^2 + \dots]\} e^{-i\sigma t}, \qquad (31a)$$
$$\overline{Y_1^{(2)}(t)}^t = 2\pi\rho g H_0^2 e^{-2\nu h}\{(\nu a)\,\mathrm{I}_1(2\nu a) - (\nu a)^4(\gamma_{11} + \gamma_{21}) + \dots\}, \qquad (31b)$$

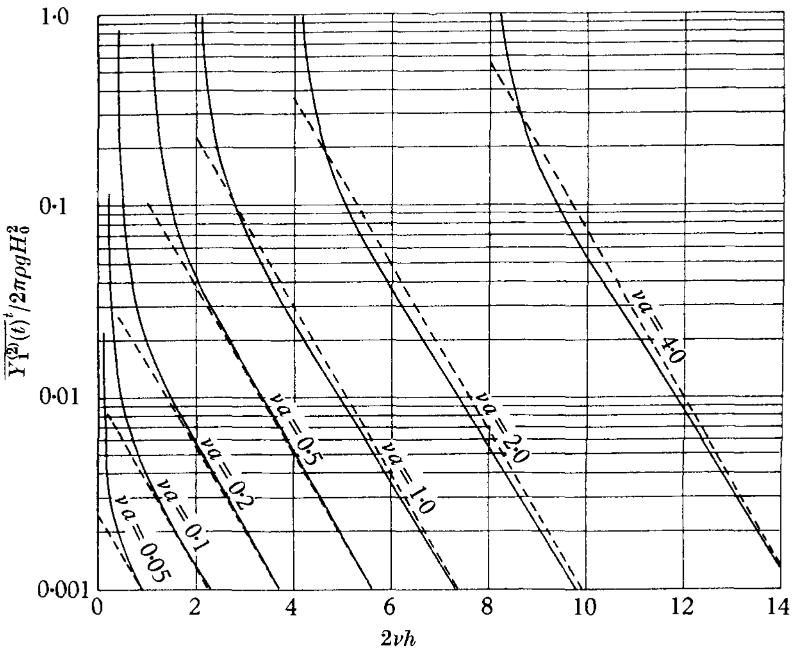

FIGURE 4. Steady vertical force on restrained cylinder.

where $\mathrm{I}_1(2\nu a)$ is the modified Bessel function of the first kind. In the last equation, all coefficients of $(\nu a)^{2n}$ after the Bessel-function term approach zero as $2\nu h \to \infty$. Thus the first term in brackets provides an approximation if *either* of the following conditions is satisfied: $\nu a \ll 1$ or $2\nu h \gg 1$.

5. Cylinder forced to oscillate

In this case we assume that ξ_1, ξ_2, η_1 and η_2 are known (i.e. given) and that there are no incident waves. The complex potential is given by (19) with $f_0(z,t) \equiv 0$. The unknown coefficients in (19) are found from (20), where now $A = 0$. The first-order force components are given by

$$X_2^{(1)}(t) - iY_2^{(1)}(t) = (2\pi\rho\sigma/\nu)\{(\delta_1 \sin\sigma t - \gamma_1\cos\sigma t) - i(\alpha_1\cos\sigma t - \beta_1\sin\sigma t)\}$$
$$-\pi\rho\sigma^2 a^2\{(\xi_1\sin\sigma t + \xi_2\cos\sigma t) - i(\eta_1\sin\sigma t + \eta_2\cos\sigma t)\}. \qquad (32)$$

The second-order steady force components are given by

$$\overline{X_2^{(2)}(t)}^t - i\overline{Y_2^{(2)}(t)}^t = -i\pi\rho\nu\sum_1^\infty \frac{m(m+1)}{(\nu a)^{2m+2}}[(\alpha_m + i\gamma_m)(\alpha_{m+1} - i\gamma_{m+1})$$
$$+ (\beta_m + i\delta_m)(\beta_{m+1} - i\delta_{m+1})]. \qquad (33)$$

It is shown in Appendix B that, if ζ_m and S_ζ are defined by

$$\zeta_m + (\nu a)^{2m} \sum_{n=1}^{\infty} \frac{(m+n)!}{m!\,(n-1)!} A_{m+n}\zeta_n = \delta_{m1}, \tag{34}$$

$$S_\zeta = 2\pi\, e^{-2\nu h} \sum_1^{\infty} \frac{\zeta_n}{(n-1)!}, \tag{34'}$$

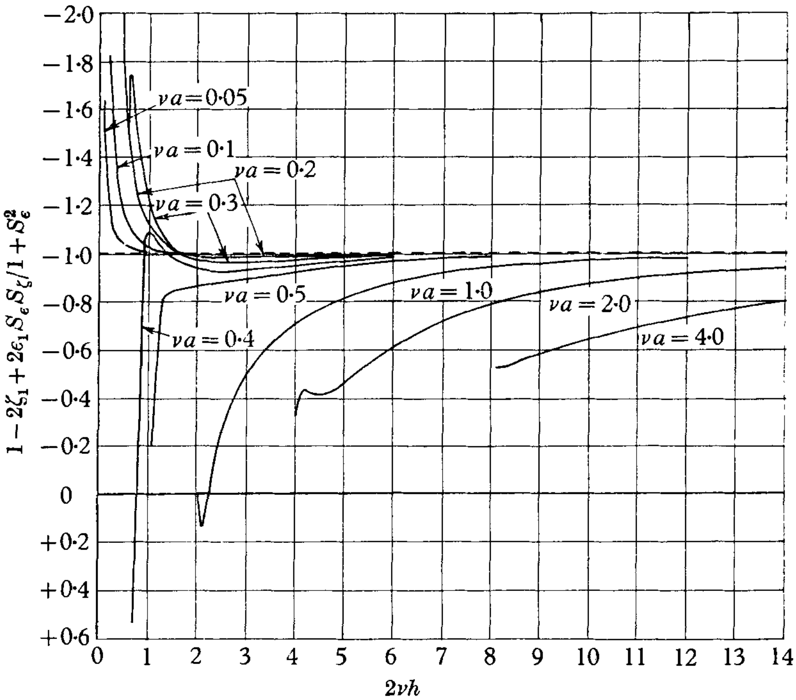

FIGURE 5. Oscillatory force on cylinder moving sinusoidally: component in phase with cylinder acceleration.

then α_m, β_m, γ_m and δ_m can be expressed in terms of ϵ_m, ζ_m and of ξ_1, ξ_2, η_1, η_2 (see equations (62)). One obtains for the force components

$$X_2^{(1)}(t) - iY_2^{(1)}(t) = -\pi\rho a^2\left\{\left[2\zeta_1 - \frac{2S_\epsilon S_\zeta \epsilon_1}{1+S_\epsilon^2} - 1\right][\ddot{\xi}^{(1)}(t) - i\ddot{\eta}^{(1)}(t)] + \left[\frac{2\sigma S_\zeta \epsilon_1}{1+S_\epsilon^2}\right][\dot{\xi}^{(1)}(t) - i\dot{\eta}(t)]\right\}; \tag{35}$$

$$\overline{X_2^{(2)}(t)}^t = \frac{2\pi\rho g S_\zeta(\xi_2\eta_1 - \xi_1\eta_2)}{1+S_\epsilon^2} \sum_1^{\infty} \frac{m(m+1)\,(\epsilon_{m+1}\zeta_m - \epsilon_m\zeta_{m+1})}{(\nu a)^{2m-2}}, \tag{36a}$$

$$\overline{Y_2^{(2)}(t)}^t = \frac{\pi\rho g(\xi_1^2+\xi_2^2+\eta_1^2+\eta_2^2)}{1+S_\epsilon^2} \sum_1^{\infty} \frac{m(m+1)}{(\nu a)^{2m-2}} \times \{\zeta_m\zeta_{m+1} + (S_\epsilon\zeta_m - S_\zeta\epsilon_m)(S_\epsilon\zeta_{m+1} - S_\zeta\epsilon_{m+1})\}. \tag{36b}$$

It is interesting to note that the horizontal steady force vanishes if the cylinder oscillates along a straight line, no matter what the orientation of this line.

Figures 5, 6 and 7 show numerical results obtained from equations (35) and (36*b*). In particular, figure 5 presents the component of hydrodynamic force which is in phase with the acceleration. If we define *added mass* as the negative ratio of this force component to the acceleration, we observe that negative added masses

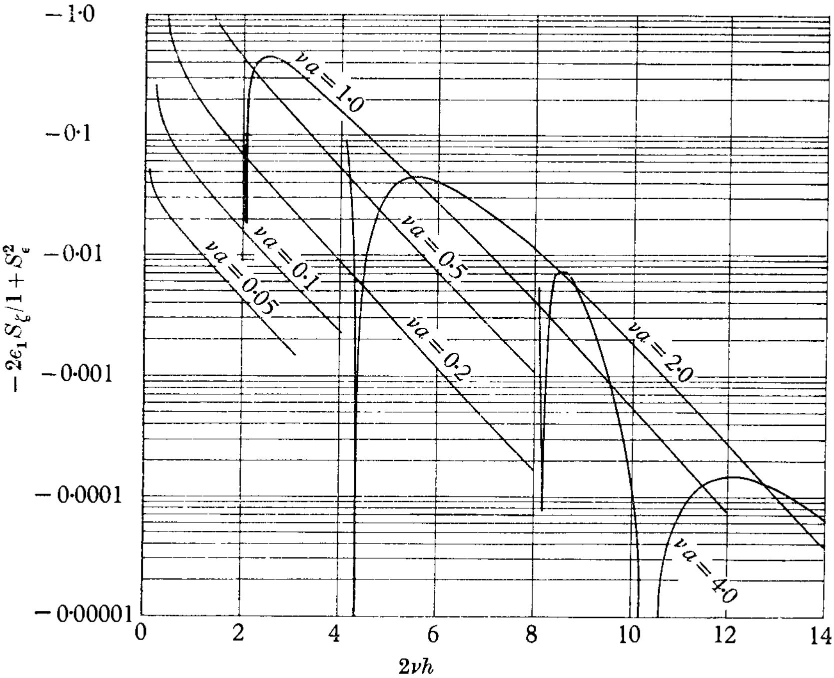

FIGURE 6. Oscillatory force on cylinder moving sinusoidally: component in phase with cylinder velocity.

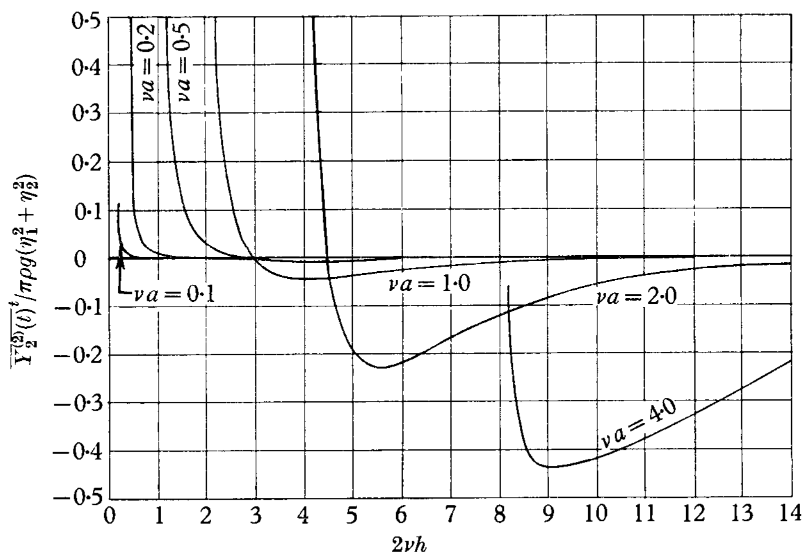

FIGURE 7. Steady vertical force on cylinder moving sinusoidally vertically.

exist under a few conditions where the cylinder is very close to the free surface. Figure 6 presents the force component which is in phase with the velocity. The ratio of this force component to the velocity is the negative of the conventional damping coefficient. Of course, this coefficient is always positive, although its value is very small for certain small values of the depth. At large values of the depth, the damping force naturally approaches zero, since the wave-making capability of any oscillating body vanishes as the depth becomes very large. Figure 7 presents the steady vertical force on the oscillating cylinder. For any value of νa, the force is upwards for small submergence, becomes negative for larger submergence, and approaches zero from below as the submergence becomes infinite. For very small values of νa, it can be shown analytically that the curves all cross zero at approximately $2\nu h = 2{\cdot}8$. This is confirmed by the calculations, although it is not apparent from the figure because of the scale of the ordinates.

6. Cylinder free to respond to waves

With the incident waves given by (17), we can write the potential as the sum of the potentials found in §§ 4 and 5, but with ξ_1, ξ_2, η_1, η_2 now unknown. The first-order force is the sum of the two forces found previously, but of course this is not true for the non-linear second-order force. Because of this last fact, it is just as convenient to return to the formulation of § 3 for solving this problem.

The complex potential is still given by (19′), with the unknown coefficients to be found by solving (20). Such solutions for the coefficients are expressed in terms of ξ_1, ξ_2, η_1, η_2—also still unknown, and the first-order force is then found in terms of these motion parameters:

$$X_3^{(1)}(t) - iY_3^{(1)}(t) = \pi\rho\sigma a\{\sin\sigma t[2(\delta_1 + i\beta_1)/\nu a - \sigma a(\xi_1 - i\eta_1)] - \cos\sigma t[2(\gamma_1 + i\alpha_1)/\nu a + \sigma a(\xi_2 - i\eta_2)]\}. \quad (37)$$

(See also equations (62) in Appendix B.) This complex force is set equal to the corresponding inertial reaction giving the equation of motion:

$$\pi\rho a^2[\ddot{\xi}^{(1)}(t) - i\ddot{\eta}^{(1)}(t)] = X_3^{(1)}(t) - iY_3^{(1)}(t). \quad (38)$$

It is easily seen that this equation requires that

$$\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0. \quad (39)$$

From Appendix B then,

$$\xi_1 = -\eta_2 = -\frac{A\,e^{-\nu h}\,\epsilon_1(S_\zeta\epsilon_1 - S_\epsilon\zeta_1)}{\sigma a(\nu a)\,[\zeta_1^2 + (S_\zeta\epsilon_1 - S_\epsilon\zeta_1)^2]}, \quad (40a)$$

$$\xi_2 = \eta_1 = \frac{A\,e^{-\nu h}\,\epsilon_1\zeta_1}{\sigma a(\nu a)\,[\zeta_1^2 + (S_\zeta\epsilon_1 - S_\epsilon\zeta_1)^2]}. \quad (40b)$$

These results can then be used in the expressions for α_m, β_m, γ_m and δ_m, equations (62), for $m > 1$, yielding

$$\alpha_m = -\delta_m = \frac{A\,e^{-\nu h}\,(S_\epsilon\zeta_1 - S_\zeta\epsilon_1)\,(\zeta_1\epsilon_m - \epsilon_1\zeta_m)}{\zeta_1^2 + (S_\zeta\epsilon_1 - S_\epsilon\zeta_1)^2}, \quad (41a)$$

$$\beta_m = \gamma_m = \frac{A\,e^{-\nu h}\,\zeta_1(\zeta_1\epsilon_m - \epsilon_1\zeta_m)}{\zeta_1^2 + (S_\zeta\epsilon_1 - S_\epsilon\zeta_1)^2}. \quad (41b)$$

The second-order steady force is found to have components

$$\overline{X_3^{(2)}(t)}^t = 0; \tag{42a}$$

$$\overline{Y_3^{(2)}(t)}^t = \frac{2\pi\rho\nu A^2 e^{-2\nu h}}{\zeta_1^2 + (S_\zeta \epsilon_1 - S_\epsilon \zeta_1)^2} \sum_1^\infty \frac{m(m+1)}{(\nu a)^{2m+2}} (\zeta_1 \epsilon_m - \epsilon_1 \zeta_m)(\zeta_1 \epsilon_{m+1} - \epsilon_1 \zeta_{m+1}). \tag{42b}$$

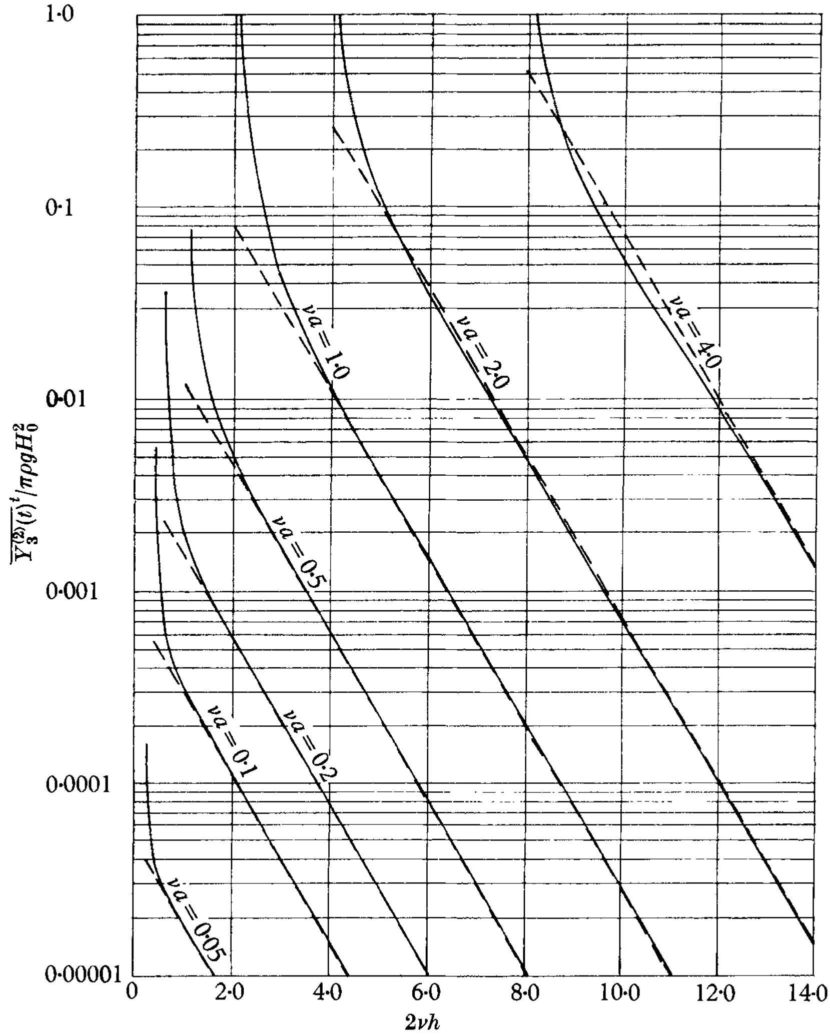

FIGURE 8. Steady vertical force on a free neutrally buoyant cylinder under waves.

We note again that $\nu A^2 = gH_0^2$, where H_0 is the amplitude of the incident surface wave. For small values of νa, this result can be approximated

$$\overline{Y_3^{(2)}(t)}^t = 2\pi\rho g H_0^2 e^{-2\nu h}\left\{\nu a[\mathrm{I}_1(2\nu a) - \nu a] + O\left[\frac{(\nu a)^8}{(2\nu h)^4}\right]\right\}. \tag{43}$$

Again, this is a good approximation if either νa is small or $2\nu h$ is large.

Figure 8 shows the results of calculations from equation (42 *b*). The steady force in this case is always upward, as in the case of the restrained cylinder under waves. If νa is very small, we note that the steady force is much smaller than was the case for the restrained cylinder. However, for larger values of νa, there is not much difference between the two cases. The approximate expressions for the steady forces, equations (31 *b*) and (43), show that this is reasonable if $2\nu h$ is large.

It is of some interest to consider the motion of the cylinder further. Clearly, from (40),

$$\xi^{(1)}(t) = \eta^{(1)}\left(t+\frac{\pi}{2\sigma}\right),$$

and so the cylinder follows a circular path. In fact,

$$\xi^{(1)}(t)-i\eta^{(1)}(t) = \frac{H_0\epsilon_1 e^{-\nu h}}{(\nu a)^2\{\zeta_1^2+(S_\epsilon\zeta_1-S_\zeta\epsilon_1)^2\}^{\frac{1}{2}}}\exp\{-i(\sigma t-\psi_2)\},$$

where

$$\psi_2 = \tan^{-1}\left\{\frac{(S_\epsilon\zeta_1-S_\zeta\epsilon_1)}{\zeta_1}\right\}.$$

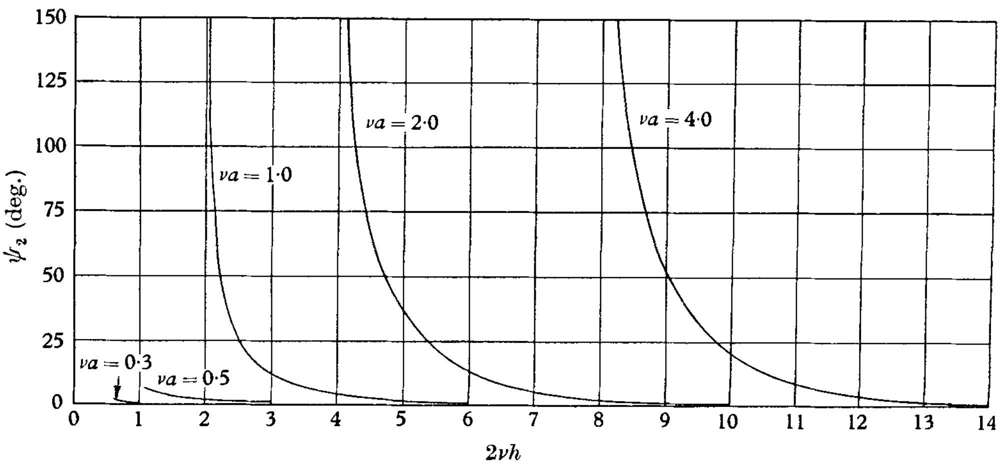

FIGURE 9. Phase lag of free cylinder motion.

For small νa, the amplitude can be approximated

$$\frac{H_0\epsilon_1 e^{-\nu h}}{(\nu a)^2\{\zeta_1^2+(S_\epsilon\zeta_1-S_\zeta\epsilon_1)^2\}^{\frac{1}{2}}} = H_0 e^{-\nu h}\{1+\gamma_{12}(\nu a)^4+O[(\nu a)^6]\}.$$

The corresponding water particle (in the absence of the cylinder) would have an orbit given by

$$\xi^{(1)}(t)-i\eta^{(1)}(t) = H_0 e^{-\nu h} e^{-i\sigma t}.$$

Thus the amplitude of the cylinder orbit differs from that of the water particle by a quantity of fourth order in νa. The phase of the cylinder motion lags behind that of the corresponding water particle by the angle

$$\psi_2 = \tan^{-1}[\pi e^{-2\nu h}(\nu a)^4+\ldots],$$

again a quantity of fourth order in νa. The quantity ψ_2 is plotted in figure 9.

Two points may be mentioned with respect to this description of the motion. (1) Since there is generally a steady upward force, the cylinder should accelerate vertically. In such a case it is impossible to formulate a steady-motion problem at all, and so we have assumed that by some artificial means the cylinder is restrained from responding to this force. (2) In a second-order theory, there is a steady drift of water in the direction of wave propagation, and this could be expected to produce a steady force which would accelerate the cylinder horizontally. However, such a force must be of higher order than second, so that after it operates for an infinite time it will have produced a steady translational velocity of the cylinder. Thus, strictly, a description of the cylinder motion should include a steady second-order horizontal velocity, but its absence does not affect the other quantities calculated.

7. Approximate solution

(a) Restrained cylinder

If there were no free surface present, Milne-Thomson's (1960) 'circle theorem' could be used to find the change caused by introducing a restrained circular cylinder into the externally-produced potential flow. If, in the absence of the cylinder, the complex potential is $A\,e^{-i(\nu z+\sigma t)}$, then the flow with the restrained cylinder present is given by the complex potential

$$f_1(z,t) = A\,e^{-i(\nu z+\sigma t)} + A\,e^{-\nu h}\exp\{i[\sigma t+\nu a^2/(z+ih)]\}, \tag{44}$$

according to the circle theorem. This potential does not satisfy the free-surface condition, but since the second term becomes unimportant if either a is small or h is large, one might expect it to yield reasonable results over an appreciable range of a and h. In this section, such approximate solutions are used to calculate the forces and a comparison is made with the theory previously developed.

As before, we have

$$X_1^{(1)}(t) - i\,Y_1^{(1)}(t) = -ia\rho\int_{-\pi}^{\pi} e^{i\theta}\,\Phi_t^{(1)}|_{r=a}\,d\theta,$$

where now

$$\Phi_t^{(1)}|_{r=a} = \mathrm{Re}\{Ai\sigma\,e^{-\nu h}[-\exp(\nu a\,e^{-i\theta})\,e^{-i\sigma t} + \exp(\nu a\,e^{i\theta})\,e^{i\sigma t}]\}.$$

Then

$$X_1^{(1)}(t) - i\,Y_1^{(1)}(t) = -2\pi\rho\sigma a(\nu a)\,A\,e^{-\nu h}\,e^{-i\sigma t}. \tag{45}$$

Similarly,

$$\overline{X_1^{(2)}(t)}^t - i\,\overline{Y_1^{(2)}(t)}^t = -2\pi i\rho\nu A^2\,e^{-2\nu h}\,(\nu a)\,\mathrm{I}_1(2\nu a), \tag{46}$$

where $\mathrm{I}_1(2\nu a)$ is a modified Bessel function of the first kind. So finally, in terms of the incident wave amplitude H_0, we have

$$X_1^{(1)}(t) - i\,Y_1^{(1)}(t) = -\frac{2\pi\rho g H_0\,e^{-\nu h}}{\nu}\,(\nu a)^2\,e^{-i\sigma t}, \tag{47}$$

$$\overline{X_1^{(2)}(t)}^t = 0, \quad \overline{Y_1^{(2)}(t)}^t = 2\pi\rho g H_0^2\,e^{-2\nu h}\,(\nu a)\,\mathrm{I}_1(2\nu a). \tag{48 a, b}$$

These are to be compared with (31a), (28a) and (31b), respectively. The first-order forces are correct to the lowest order in powers of νa. The second-order horizontal steady force is given correctly (zero) by the approximate solution. The second-order vertical-force expression is only approximately correct. These results are shown as broken lines on figures 2 and 4. In this approximation, the phase lag, ψ_1, presented in figure 3 is identically zero.

(b) *Cylinder forced to oscillate*

In § 7 (*a*), the mathematical expression for the effect of the cylinder is equivalent to a set of multipole potentials, the singularities being located within the cylinder. That part of the potential of § 4 which represents singularities in the upper half-plane is neglected. In other words, the effect of the free surface on the disturbance due to the cylinder is not considered. In the present case, where the only disturbance is due to the oscillating cylinder, the corresponding complex potential is

$$f_2(z,t) = -\frac{a^2}{z+ih}[\dot{\xi}^{(1)}(t) - i\dot{\eta}^{(1)}(t)]. \tag{49}$$

This is just the classical potential for a cylinder oscillating in an infinite fluid. The only first-order hydrodynamic force on the cylinder is the familiar added-mass force,

$$X_2^{(1)}(t) - iY_2^{(1)}(t) = -\pi\rho a^2[\ddot{\xi}^{(1)}(t) - i\ddot{\eta}^{(1)}(t)]. \tag{50}$$

The second-order force is easily found to be zero. These results are shown by the broken lines in figure 5. In figures 6 and 7, this approximation yields answers identically equal to zero.

(c) *Cylinder free to respond to waves*

The potential is now the sum of the potentials in the last two subsections, viz.

$$f_3(z,t) = A\,e^{-i(\nu z+\sigma t)} + A\,e^{-\nu h}\exp[i\{\sigma t + \nu a^2/(z+ih)\}] - [a^2/(z+ih)]\,[\dot{\xi}^{(1)}(t) - i\dot{\eta}^{(1)}(t)],$$

with $\xi^{(1)}(t)$ and $\eta^{(1)}(t)$ now unknown functions. To find the motion, we solve the equation

$$\begin{aligned}\pi\rho a^2(\ddot{\xi}^{(1)} - i\ddot{\eta}^{(1)}) &= [X_1^{(1)}(t) + X_2^{(1)}(t)] - i[Y_1^{(1)}(t) + Y_2^{(1)}(t)] \\ &= -2\pi\rho A\sigma a(\nu a)\,e^{-\nu h}\,e^{-i\sigma t} - \pi\rho a^2(\ddot{\xi}^{(1)} - i\ddot{\eta}^{(1)}) \\ &= -\pi\rho A\sigma a(\nu a)\,e^{-\nu h}\,e^{-i\sigma t}.\end{aligned} \tag{51}$$

We obtain

$$\xi^{(1)}(t) - i\eta^{(1)}(t) = (A\nu\,e^{-\nu h}/\sigma)\,e^{-i\sigma t} = H_0\,e^{-\nu h}\,e^{-i\sigma t}. \tag{52}$$

This is identical with the motion of the equivalent water particle in the absence of the cylinder. We find the second-order steady forces directly from (16″):

$$\overline{X_3^{(2)}(t)}^t = 0, \quad \overline{Y_3^{(2)}(t)}^t = 2\pi\rho g H_0^2\,e^{-2\nu h}\,(\nu a)\,[\mathrm{I}_1(2\nu a) - (\nu a)]. \tag{53 a, b}$$

This result is to be compared with (43). Again the first terms of the exact solution are given correctly. This approximate value is indicated on figure 8 by the broken lines.

8. Reflexion and transmission of waves

Dean and Ursell showed that in the problem of § 4 (cylinder restrained) there is no reflected wave, but that the transmitted wave has a phase shift after passing the cylinder. This phase shift is readily calculated from the quantities already involved in the force calculations. To show this we first note that the basic singularity potentials of Appendix A have the following asymptotic behaviour:

$$\operatorname{Re}\{f_{n1}(z,t)\} \xrightarrow[x\to\pm\infty]{} -\frac{2\pi}{(n-1)!}e^{\nu(y-h)}\cos(\nu x\mp\sigma t),$$

$$\operatorname{Re}\{f_{n2}(z,t)\} \xrightarrow[x\to\pm\infty]{} \mp\frac{2\pi}{(n-1)!}e^{\nu(y-h)}\sin(\nu x\mp\sigma t),$$

$$\operatorname{Re}\{g_{n1}(z,t)\} \xrightarrow[x\to\pm\infty]{} \frac{2\pi}{(n-1)!}e^{\nu(y-h)}\sin(\nu x\mp\sigma t),$$

$$\operatorname{Re}\{g_{n2}(z,t)\} \xrightarrow[x\to\pm\infty]{} \mp\frac{2\pi}{(n-1)!}e^{\nu(y-h)}\cos(\nu x\mp\sigma t).$$

These are substituted into equation (19), together with (63). Use of the definition (59) then gives for the wave shape

$$H(x,t)\to\begin{cases} H_0\sin(\nu x+\sigma t) & (x\to+\infty),\\ \dfrac{H_0}{1+S_\epsilon^2}[(1-S_\epsilon^2)\sin(\nu x+\sigma t)-2S_\epsilon\cos(\nu x+\sigma t)] & (x\to-\infty).\end{cases}$$

Thus the wave shows a phase lag of $\tan^{-1}\{2S_\epsilon/(1-S_\epsilon^2)\} = 2\psi_1$, where ψ_1 was defined in equation (27′) and plotted in figure 3.

The fact that ψ_1 seems to increase monotonely as $2\nu h$ decreases (for fixed νa) suggests an interpretation of what is happening physically. Consider the case $\nu a = 4{\cdot}0$, where the diameter-to-wavelength ratio is about 1·3. No matter how shallow the submergence, essentially all of the wave energy passes above the cylinder, none below. Thus when a wave approaches the cylinder, the situation is similar to that of a wave entering shallow water. Its phase speed is reduced, and so when it emerges on the opposite side it exhibits a phase lag compared to the undisturbed wave. The smaller the value of $(h-a)$, the greater will be the phase lag. If we suppose further that, because of the geometrical symmetry, half of the lag occurs before the wave reaches $x = 0$ and half afterwards, then it is reasonable that the oscillatory forces should show a phase lag just one-half of the total transmission phase lag. The force in a sense depends on the average phase of the passing wave.

When the value of νa is quite small, the picture is not so clear-cut, for then at small submergence some of the wave motion occurs under the cylinder. We should not expect the phase lag to increase indefinitely as $h-a\to 0$, but the argument relating force phase and transmitted-wave phase is still valid.

Next, for the problem of § 5 (cylinder oscillating in otherwise calm water), we find the amplitude of outgoing waves. In particular, for circular orbits of the cylinder, we show that progressive waves are produced in only one direction. The procedure is exactly as before. Substitute the asymptotic expansions above into (19) and use equations (62), with $A = 0$. For the velocity potential we find

$$\operatorname{Re}\{f_2(z,t)\} \xrightarrow[x\to\pm\infty]{} \sigma a S_\zeta(\nu a)\,e^{\nu(y-h)}/(1+S_\epsilon^2)\{[-\eta_2+\eta_1 S_\epsilon\mp\xi_1\mp\xi_2 S_\epsilon]\cos(\nu x\mp\sigma t)$$
$$+[\pm\eta_1\pm\eta_2 S_\epsilon-\xi_2+\xi_1 S_\epsilon]\sin(\nu x\mp\sigma t)\}.$$

If the orbit of the cylinder centre is counter-clockwise, that is,

$$\eta^{(1)}(t) \doteq \xi^{(1)}(t-\pi/2\sigma),$$

then

$$\begin{aligned}\operatorname{Re}\{f_2(z,t)\} &\xrightarrow[x\to+\infty]{} 0\\ &\xrightarrow[x\to-\infty]{} 2\sigma a S_\zeta(\nu a)\, e^{\nu(y-h)}/(1+S_\epsilon^2)\\ &\qquad \times\{(-\eta_2+\eta_1 S_\epsilon)\cos(\nu x+\sigma t)-(\eta_1+\eta_2 S_\epsilon)\sin(\nu x+\sigma t)\},\end{aligned}$$

and waves are generated only to the left. If the orbit is clockwise, that is,

$$\eta^{(1)}(t) = \xi^{(1)}(t+\pi/2\sigma),$$

then

$$\begin{aligned}\operatorname{Re}\{f_2(z,t)\} &\xrightarrow[x\to+\infty]{} 2\sigma a S_\zeta(\nu a)\, e^{\nu(y-h)}/(1+S_\epsilon^2)\\ &\qquad \times\{(-\eta_2+\eta_1 S_\epsilon)\cos(\nu x-\sigma t)+(\eta_1+\eta_2 S_\epsilon)\sin(\nu x-\sigma t)\}\\ &\xrightarrow[x\to-\infty]{} 0,\end{aligned}$$

and waves are generated only to the right.

This unilateral production of waves can be made to appear plausible in the following way. If the cylinder oscillates only vertically, then the generated waves are symmetrical in x. If the motion is only horizontal, the generated waves are anti-symmetrical in x. If now equal vertical and horizontal cylinder motions are combined, the relative phase of the two components of motion can be adjusted so that the outgoing waves on one side just cancel each other. But because of the different symmetry characteristics of the two waves, they will certainly not cancel on the other side. It is seen above that the circular cylinder paths provide just the appropriate phase differences for this condition.

Finally, in the case of the free cylinder, we find that again there are no reflected waves. The velocity potential in this case is a linear superposition of the potentials of §§ 4 and 5. At both infinities the potential of § 4 represents waves moving to the left, as already shown. The induced cylinder motion orbit is circular, and, from equations (40), the sense is seen to be counter-clockwise. According to the results above, this motion produces outgoing waves to the left only. Thus there are no outgoing waves to the right for the combined potential, which is equivalent to saying there is no reflected wave. The actual phase lag of the transmitted wave is again easily calculated. It is found to be

$$\tan^{-1}\frac{2\zeta_1(S_\epsilon\zeta_1-S_\zeta\epsilon_1)}{\zeta_1^2-(S_\epsilon\zeta_1-S_\zeta\epsilon_1)^2} = 2\psi_2,$$

where ψ_2 was already defined as the phase lag of the motion of the cylinder and was shown numerically in figure 9. Comparison of curves of ψ_2 and ψ_1 shows that for large νa there is practically no difference, but for small νa the free cylinder causes practically no phase shift compared with that caused by the restrained cylinder. This is quite reasonable. A free cylinder which is small compared to wavelength will respond to the waves very much as if it were simply made of water particles, and so it will not greatly disturb the wave motion. A large free cylinder on the other hand undergoes much less motion that the equivalent water particles, since it responds to more or less of an average pressure disturbance, and so it affects the waves in much the same manner as the restrained cylinder. A similar situation was noted in the comparison of figures 8 and 4.

Most of this work was done while the author was employed by the David Taylor Model Basin, Washington, D.C. Personally the author thanks Dr J. Nicholas Newman for valuable discussions of the subject, Mrs Patricia M. Monacella for programming and carrying out the lengthy calculations on the IBM 7090 computer, and Mr James M. Newman for checking many of the analytical results.

Appendix A

Pulsating singularity potentials

Let

$$F(z,t) = \left\{\log\frac{\nu(z+ih)}{\nu(z-ih)} - 2\mathscr{P}\int_0^\infty \frac{e^{-ik(z-ih)}}{k-\nu}\,dk\right\}\sin\sigma t + \{2\pi\, e^{-i\nu(z-ih)}\}\cos\sigma t, \quad (54a)$$

$$G(z,t) = \left\{i\log\nu^2(z+ih)(z-ih) + 2i\mathscr{P}\int_0^\infty \frac{e^{-ik(z-ih)}}{k-\nu}\,dk\right\}\sin\sigma t - \{2\pi i\, e^{-i\nu(z-ih)}\}\cos\sigma t. \quad (54b)$$

The improper integrals are to be interpreted in the Cauchy principal value sense. $\mathrm{Re}\{F(z,t)\}$ is the potential function (see Wehausen & Laitone 1960) for a source, located at $z = -ih$, which is pulsating in time with an instantaneous outflow equal to $2\pi\sin\sigma t$. The undisturbed free surface lies in the x-axis. $\mathrm{Re}\{G(z,t)\}$ is similarly the potential function of an oscillating vortex at $z = -ih$. Both potentials represent outgoing waves at right- and left-hand infinities.

Since both functions are analytic in the half-plane $y < h$ except for logarithmic singularities at $z = -ih$, the analytic portions can be expanded in Taylor series about $z = -ih$. The following expansions are obtained, valid in $|z+ih| < 2h$:

$$F(z,t) = \left\{\log\nu(z+ih) + \sum_0^\infty A_m[-i\nu(z+ih)]^m\right\}\sin\sigma t + \left\{\sum_0^\infty B_m[-i\nu(z+ih)]^m\right\}\cos\sigma t, \quad (54a')$$

$$G(z,t) = \left\{i\log\nu(z+ih) - \sum_0^\infty iA_m[-i\nu(z+ih)]^m\right\}\sin\sigma t - \left\{\sum_0^\infty iB_m[-i\nu(z+ih)]^m\right\}\cos\sigma t, \quad (54b')$$

where
$$A_m = \frac{1}{m(2\nu h)^m} + \frac{2}{m!}\left[e^{-2\nu h}\overline{Ei}(2\nu h) - \sum_{j=1}^m \frac{(j-1)!}{(2\nu h)^j}\right] \quad (m \geqslant 1), \quad (55a)$$

$$A_0 = -\log(-2i\nu h) + 2\,e^{-2\nu h}\,\overline{Ei}(2\nu h); \quad (55a')$$

$$B_m = \frac{2\pi\, e^{-2\nu h}}{m!}; \quad (55b)$$

$$Ei(2\nu h) = \mathscr{P}\int_{-\infty}^{2\nu h}\frac{e^u\,du}{u}.$$

If either $F(z,t)$ or $G(z,t)$ is differentiated n times with respect to z, the real part of the resulting expression still satisfies the same boundary conditions, viz. the free-surface condition and the condition that only outgoing waves exist at infinity. After relabelling these derivatives, multiplying by some real constants, and adding complex potentials in time quadrature to these, we obtain the following basic sets of singularity potentials:

$$f_{n1}(z,t) = \left\{\frac{e^{in\theta}}{(\nu r)^n} - \sum_{m=0}^{\infty} A_{m+n}\frac{(m+n)!}{m!\,(n-1)!}(\nu r)^m e^{-im\theta}\right\}\sin\sigma t$$
$$-\left\{\sum_{m=0}^{\infty} B_{m+n}\frac{(m+n)!}{m!\,(n-1)!}(\nu r)^m e^{-im\theta}\right\}\cos\sigma t; \quad (56a)$$

$$f_{n2}(z,t) = \left\{\frac{e^{in\theta}}{(\nu r)^n} - \sum_{m=0}^{\infty} A_{m+n}\frac{(m+n)!}{m!\,(n-1)!}(\nu r)^m e^{-im\theta}\right\}\cos\sigma t$$
$$+\left\{\sum_{m=0}^{\infty} B_{m+n}\frac{(m+n)!}{m!\,(n-1)!}(\nu r)^m e^{-im\theta}\right\}\sin\sigma t; \quad (56b)$$

$$g_{n1}(z,t) = \left\{\frac{i\,e^{in\theta}}{(\nu r)^n} + i\sum_{m=0}^{\infty} A_{m+n}\frac{(m+n)!}{m!\,(n-1)!}(\nu r)^m e^{-im\theta}\right\}\sin\sigma t$$
$$+\left\{\sum_{m=0}^{\infty} iB_{m+n}\frac{(m+n)!}{m!\,(n-1)!}(\nu r)^m e^{-im\theta}\right\}\cos\sigma t; \quad (56c)$$

$$g_{n2}(z,t) = \left\{\frac{i\,e^{in\theta}}{(\nu r)^n} + i\sum_{m=0}^{\infty} A_{m+n}\frac{(m+n)!}{m!\,(n-1)!}(\nu r)^m e^{-im\theta}\right\}\cos\sigma t$$
$$-\left\{\sum_{m=0}^{\infty} iB_{m+n}\frac{(m+n)!}{m!\,(n-1)!}(\nu r)^m e^{-im\theta}\right\}\sin\sigma t. \quad (56d)$$

$(z+ih)$ has been expressed in polar co-ordinates as

$$z+ih = r\,e^{i(\frac{1}{2}\pi-\theta)} = ir\,e^{-i\theta}. \quad (57)$$

It is of interest to note the form of $A_m(2\nu h)$ for large values of $2\nu h$. As $2\nu h \to \infty$,

$$e^{-2\nu h}\,\overline{Ei}(2\nu h) - \sum_{j=1}^{m}\frac{(j-1)!}{(2\nu h)^j} = \sum_{j=m+1}^{N}\frac{(j-1)!}{(2\nu h)^j} + o\left(\frac{1}{(2\nu h)^N}\right),$$

where N is any integer larger than $(m+1)$ (see Jahnke & Emde 1945). Thus, asymptotically,

$$A_m = \frac{1}{m!}\left\{\frac{(m-1)!}{(2\nu h)^m} + 2\sum_{j=m+1}^{N}\frac{(j-1)!}{(2\nu h)^j} + o\left(\frac{1}{(2\nu h)^N}\right)\right\},$$

as $2\nu h \to \infty$.

Appendix B *Solution of infinite sets of equations*

As shown by Ursell, the four sets of equations, (20), can be effectively uncoupled. Let

$$\gamma_{mn} = ((m+n)!/(n-1)!)\,A_{m+n}; \quad (58)$$

$$\delta_{mn} = \begin{cases}1 & \text{if } m = n,\\ 0 & \text{if } m \neq n;\end{cases}$$

$$S_x = 2\pi\,e^{-2\nu h}\sum_{1}^{\infty}\frac{x_n}{(n-1)!}, \quad (59)$$

where $\{x_n\}$ is any sequence such that this sum exists. From Appendix A,

$$B_{m+n} = \frac{2\pi\,e^{-2\nu h}}{(m+n)!}.$$

Thus the unknown coefficients satisfy

$$\alpha_m + \frac{(\nu a)^{2m}}{m!} \sum_{n=1}^{\infty} \gamma_{mn} \alpha_n = \frac{(\nu a)^{2m}}{m!} S_\beta + \eta_2 \sigma a(\nu a)\, \delta_{m1};$$

$$\beta_m + \frac{(\nu a)^{2m}}{m!} \sum_{n=1}^{\infty} \gamma_{mn} \beta_n = -\frac{(\nu a)^{2m}}{m!} S_\alpha - \eta_1 \sigma a(\nu a)\, \delta_{m1} + \frac{A\, e^{-\nu h}\, (\nu a)^{2m}}{m!};$$

$$\gamma_m + \frac{(\nu a)^{2m}}{m!} \sum_{n=1}^{\infty} \gamma_{mn} \gamma_n = \frac{(\nu a)^{2m}}{m!} S_\delta - \xi_2 \sigma a(\nu a)\, \delta_{m1} + \frac{A\, e^{-\nu h}\, (\nu a)^{2m}}{m!};$$

$$\delta_m + \frac{(\nu a)^{2m}}{m!} \sum_{n=1}^{\infty} \gamma_{mn} \delta_n = -\frac{(\nu a)^{2m}}{m!} S_\gamma + \xi_1 \sigma a(\nu a)\, \delta_{m1}.$$

Let ϵ_m and ζ_m be defined by

$$\epsilon_m + \frac{(\nu a)^{2m}}{m!} \sum_{n=1}^{\infty} \gamma_{mn} \epsilon_n = \frac{(\nu a)^{2m}}{m!}; \tag{60a}$$

$$\zeta_m + \frac{(\nu a)^{2m}}{m!} \sum_{n=1}^{\infty} \gamma_{mn} \zeta_n = \delta_{m1}. \tag{60b}$$

Then, if the set of homogeneous equations corresponding to any of these has only zero solutions (which is generally the case), linear combinations of the above six sets of equations can be formed to give the results:

$$\alpha_m = S_\beta \epsilon_m + \eta_2 \sigma a(\nu a)\, \zeta_m; \tag{61a}$$

$$\beta_m = (-S_\alpha + A\, e^{-\nu h})\, \epsilon_m - \eta_1 \sigma a(\nu a)\, \zeta_m; \tag{61b}$$

$$\gamma_m = (S_\delta + A\, e^{-\nu h})\, \epsilon_m - \xi_2 \sigma a(\nu a)\, \zeta_m; \tag{61c}$$

$$\delta_m = -S_\gamma \epsilon_m + \xi_1 \sigma a(\nu a)\, \zeta_m. \tag{61d}$$

We then form the sums, S_α, etc., by (59), which provides four linear algebraic equations in S_α, S_β, S_γ and S_δ, which we can solve. The results are then substituted into the above expressions for α_m, β_m, γ_m and δ_m, yielding

$$\alpha_m = -\frac{\sigma a S_\zeta(\nu a)\,(\eta_1 + \eta_2 S_\epsilon) + A\, e^{-\nu h}\, S_\epsilon}{1 + S_\epsilon^2}\, \epsilon_m + \eta_2 \sigma a(\nu a)\, \zeta_m; \tag{62a}$$

$$\beta_m = \frac{\sigma a S_\zeta(\nu a)\,(-\eta_2 + \eta_1 S_\epsilon) + A\, e^{-\nu h}}{1 + S_\epsilon^2}\, \epsilon_m - \eta_1 \sigma a(\nu a)\, \zeta_m; \tag{62b}$$

$$\gamma_m = \frac{\sigma a S_\zeta(\nu a)\,(\xi_1 + \xi_2 S_\epsilon) + A\, e^{-\nu h}}{1 + S_\epsilon^2}\, \epsilon_m - \xi_2 \sigma a(\nu a)\, \zeta_m; \tag{62c}$$

$$\delta_m = \frac{\sigma a S_\zeta(\nu a)\,(\xi_2 - \xi_1 S_\epsilon) - A\, e^{-\nu h}\, S_\epsilon}{1 + S_\epsilon^2}\, \epsilon_m + \xi_1 \sigma a(\nu a)\, \zeta_m. \tag{62d}$$

In the special case of no cylinder motion, these results simplify greatly to

$$\alpha_m = -\delta_m = \frac{A\, e^{-\nu h}\, S_\epsilon}{1 + S_\epsilon^2}\, \epsilon_m; \tag{63a}$$

$$\beta_m = \gamma_m = \frac{A\, e^{-\nu h}}{1 + S_\epsilon^2}\, \epsilon_m \tag{63b}$$

(see equation (21)).

The sets of equations (60*a*) and (60*b*) were truncated for solution. First they were each cut off with only ten unknowns and ten equations and the desired forces were all calculated. Then the procedure was repeated with twenty equations and twenty unknowns and the forces recalculated. In cases where h/a was only slightly greater than one, there was generally some discrepancy, and then the procedure was repeated again with forty equations and forty unknowns. For the results reported in the figures of this paper, the last two calculations agreed to at least three significant figures.

REFERENCES

DEAN, W. R. 1948 On the reflection of surface waves by a submerged cylinder. *Proc. Camb. Phil. Soc.* **44**, 483–91.

JAHNKE, E. & EMDE, F. 1945 *Tables of Functions.* New York: Dover.

MILNE-THOMSON, L. M. 1960 *Theoretical Hydrodynamics*, 4th ed. New York: MacMillan.

STOKER, J. J. 1957 *Water Waves.* New York: Interscience.

URSELL, F. 1950 Surface waves on deep water in the presence of a submerged circular cylinder. I. *Proc. Camb. Phil. Soc.* **46**, 141–52

WEHAUSEN, J. V. & LAITONE, E. V. 1960 Surface waves. *Handbuch der Physik,* Vol. IX. Berlin: Springer-Verlag.